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GALOIS GROUPS AND COMPLETE DOMAINS

BY

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ABSTRACT

Consider a domain $\hat{R}$ that is complete with respect to a non-zero prime ideal. This paper proves two Galois-theoretic results about such rings. Using Grothendieck's Existence Theorem we prove that every finite group occurs as the Galois group of a Galois extension of $\hat{R}[x]$. This generalizes results of David Harbater who proved the result in the case where the ideal is maximal and the domain is normal. As a consequence, we deduce that if $\hat{R}$ is a Noetherian domain that is complete with respect to a non-zero prime ideal, then every finite group occurs as a Galois group over $\hat{R}$. This proves the Noetherian case of a conjecture posed by Moshe Jarden.

1. Introduction

Let $\mathbb{A}^1_R$ be $\mathrm{Spec}(R[t])$, the affine t-line over a ring R. Let $\mathbb{P}^1_R$ be the projective t-line. Using Riemann's Existence Theorem one can construct algebraic branched covers of the complex projective line $\mathbb{P}^1_{\mathbb{C}}$ with arbitrary (finite) Galois group. Using techniques from formal geometry, Harbater [H3] constructs branched covers of the p-adic projective line $\mathbb{P}^1_{\mathbb{Q}_p}$ with arbitrary Galois group, and more generally shows every finite group occurs as a Galois group over the fraction field of $\hat{R}[t]$, where $\hat{R}$ is a normal domain that is complete with respect to a maximal ideal $\mathfrak{m}$. Liu [Li] shows these results from the point of view of rigid geometry. Further discussion of these results can be found in [V] and [MMa]. This paper generalizes these results.

Now let $\hat{R}$ be a domain containing a domain (not necessarily normal) that is complete at a non-zero prime ideal (not necessarily maximal). We prove that

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every finite group G is the Galois group of a regular cover of $\mathbb{P}^1_{\hat{R}}$. Thus G is the Galois group of some Galois extension of $\hat{R}[x]$ (and its fraction field) (Theorem 3.17 and Corollary 3.18). Using Grothendieck's Existence Theorem, we first prove the special case of $\hat{R} = \mathbb{Z}[[x]]$ by adapting arguments from [H2] and [H3]. We then use this special case and results from [H3] to extend to the general case.

As a corollary, we deduce that if $\hat{R}$ is a Noetherian domain of dimension at least two which contains a domain that is complete with respect to a non-zero prime ideal, then every finite group occurs as a Galois group over $\hat{R}$ (Corollary 3.20). As a consequence we prove the Noetherian case of a conjecture posed by Moshe Jarden; namely, if $\hat{R} = D[[x_1, ..., x_r]]$ is a power series ring of dimension at least two with coefficients in a Noetherian domain D, then every finite group occurs as a Galois group over $\hat{R}$ (Corollary 3.21).

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2. Definitions, notation and terminology

This section introduces the notation and terminology used throughout the paper. The reader may wish to skip this section on a preliminary reading, and refer back to it as necessary.

Section 3 proves results in algebraic geometry and commutative algebra. We first introduce the algebro-geometric concepts used in this paper and include some basic remarks.

Definition 2.1: If a scheme Z is a minimal finite union of irreducible schemes $Z_1, \ldots, Z_n$, then each Z_i is an **irreducible component** or a **sheet** of Z. If $Z = \mathrm{Spec}(A)$, and the zero ideal $(0) \subset A$ has a primary decomposition, then the irreducible components of Z correspond to the primary decomposition of the zero ideal. Thus a sufficient condition for a scheme to have finitely many irreducible components is for it to be Noetherian.

Definition 2.2: Let A be a domain. An extension of A-algebras $A \subseteq B$ is **generically separable** if the total ring of fractions of B is separable over $\mathrm{Frac}(A)$ and no non-zero element of A becomes a zero-divisor in B [H4, p. 493]. A morphism of affine schemes $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is **generically separable** if the corresponding extension of A-algebras is generically separable. A morphism of schemes $Y \to X$ is **generically separable** if it is generically separable on each affine open subset of X.

Definition 2.3: Let X be an integral scheme and L a proper closed subset of X. A **cover** of X with **branch locus** L is a generically separable (Definition 2.2) finite morphism π: $Z \to X$ which is étale over $U = X - L$. Such a cover is a **mock cover** if the restriction of π to each irreducible component (Definition 2.1) of Z is an isomorphism [H1, p. 403].

Definition 2.4: Given a finite, separable extension of fields $K(x) \subset F$ we define the **corresponding cover** $X \to \mathbb{P}^1_k$ to be the cover (Definition 2.3) of the projective t-line over K obtained by taking the normalization of $\mathbb{P}^1_K$ in F.

Definition 2.5: Given a domain $\hat{R}$ that is complete with respect to a non-zero prime ideal $\mathfrak{a}$ and given proper morphisms $X \to \mathrm{Spec}(\hat{R})$, $Y \to X$, we define the **central fibre** of $Y \to X$ to be the fibre over $\mathfrak{a}$, i.e. the morphism $Y \times_{\mathrm{Spec}(\hat{R})} \mathrm{Spec}(\hat{R}/\mathfrak{a}) \to X \times_{\mathrm{Spec}(\hat{R})} \mathrm{Spec}(\hat{R}/\mathfrak{a})$.

Notation 2.6: If the central fibre (Definition 2.5) of a cover (Definition 2.3) $Y \to X$ is a mock cover, we will say that this cover is **mock on the central fibre**.

Definition 2.7: We say that a scheme X is **irreducible** if its underlying topological space is irreducible [Ha, II.3, p. 82]. We say that a scheme X is **locally irreducible** if for each point $x \in X$, the spectrum $\mathrm{Spec}(\mathcal{O}_{X,x})$ of the local ring at x is irreducible. It follows that if X is irreducible then X is locally irreducible.

Remark 2.8: (a) Let X be a scheme such that $\mathrm{Sp}(X)$ is locally finite (e.g. if X is locally Noetherian). If X is locally irreducible (Definition 2.7) and connected, then it is irreducible. This is seen as follows: By replacing X with X_{red}, if necessary, we may assume that X is reduced. It follows from [Ha, II, Exercise 2.3] that the local ring of the reduced scheme at a point x is the reduction of the local ring of X at x. Thus, by [Ha, II, Prop. 3.1] we may assume that the local rings are integral. It follows from [Gr1, Chap. 1, Cor. 4.5.6] that X is integral and hence irreducible.

(b) If X is normal, then X is locally irreducible. Hence in this case, X is irreducible if and only if X is connected.

Definition 2.9: Let X be a reduced and irreducible scheme. Let $Y \to X$ be a morphism of schemes. Let $\mathrm{Spec}(A)$ be an affine open subset of X. Let $K = K(X) = \mathrm{Frac}(A)$ be the field of functions on X. Then a **generic point** of X is a morphism ϕ: $\mathrm{Spec}(K) \to X$. A **generic fibre** of the morphism $Y \to X$ is the pullback of ϕ along $Y \to X$. Let $\bar{K}$ be the algebraic closure of K. Then the extension of fields $K \subseteq \bar{K}$ corresponds to a morphism ψ: $\mathrm{Spec}(\bar{K}) \to \mathrm{Spec}(K)$.

The composition $\phi\psi$ is a **generic geometric point** of X. The pullback of the composition along $Y \to X$ is a **generic geometric fibre** of the morphism $Y \to X$.

Definition 2.10: Given a finite group G, a G-**Galois cover** (or more briefly, a G-**cover**) of schemes is a cover (Definition 2.3) $Y \to X$ together with a group homomorphism $G \to \mathrm{Aut}_X Y$ which induces a simply transitive action of G on a generic geometric fibre (Definition 2.9) [H2, p. 281]. If $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$ are affine, then we say the corresponding extension of A-algebras $A \subset B$, together with the induced homomorphism $G \to \mathrm{Aut}_A B$, is a G-**Galois extension of domains**. For example: A finite separable extension of a field $K(x) \subset L$ is G-Galois if the corresponding cover (Definition 2.4) is G-Galois. The extension $K(x) \subset L$ is taken with the induced homomorphism $G \to \mathrm{Aut}_{K(x)} L$. Note that we allow ramification in a G-Galois extension of domains.

Definition 2.11: Let K be a field and let $K \subseteq L$ be an extension of K. We call this extension **regular** if it is separable and if K is algebraically closed in L.

Definition 2.12: Let $\mathrm{Spec}(S) \to \mathbb{A}^1_R$ be a cover (Definition 2.3). We will call the cover a **regular cover of** $\mathbb{A}^1_R$ if $\mathrm{Frac}(R)$ is algebraically closed in $\mathrm{Frac}(S_i)$ for each S_i, where S_i is the domain corresponding to an irreducible component of $\mathrm{Spec}(S)$ and i runs over the irreducible components of $\mathrm{Spec}(S)$. Note that by the definition of a cover, $\mathrm{Frac}(R) \subset \mathrm{Frac}(S_i)$ is a separable extension of fields, hence $\mathrm{Frac}(R) \subset \mathrm{Frac}(S_i)$ a regular extension of fields (Definition 2.11). We will call a cover of $\mathbb{P}^1_R$ a **regular cover** if it is regular on any (and hence every) affine patch.

Definition 2.13: Following [Ja], say that a group G is **regular over a field** K if G is the Galois group of a Galois field extension $K(x) \subset F$ with K algebraically closed in F. By definition of Galois, this field extension is separable; hence F/K is a regular field extension. We say that a group G is **regular over K with a rational point** if G is the Galois group of a Galois extension of $K(x) \subset F$ and the corresponding cover has a simple K-rational point. In this case G is indeed regular over K (see Lemma 3.1). We say that that a group G is **regular over a domain D (with a rational point)** if the corresponding statement holds over $\mathrm{Frac}(D)$.

Definition 2.14: We say that a finite group G is **realizable** over a domain R, if there is a G-Galois extension of domains (Definition 2.10) $R \subset S$ with Galois group G.

Definition 2.15: Let $R \subset R'$ be an extension of domains. Let $X \to \mathbb{P}^1_{R'}$ be a cover (Definition 2.3) with branch locus $L' \subset \mathbb{P}^1_{R'}$. We say the branch locus of $X \to \mathbb{P}^1_{R'}$ is **defined over** R if there is a closed subset $L \subset \mathbb{P}^1_R$ such that L' is the pullback of L with respect to $\mathbb{P}^1_{R'} \to \mathbb{P}^1_R$.

The results in Section 3 use techniques from formal algebraic geometry. We introduce some basic concepts from formal geometry from [Ha, II.9].

Remark 2.16: The theory of schemes in algebraic geometry is richer than the theory of varieties. Unlike the field of functions on a variety, the structure sheaf of a scheme may contain nilpotent elements. If Y is a subscheme of X defined by the sheaf of ideals $\mathcal{I}$, we can define the (non-reduced) scheme Y_n as the subscheme defined by the n-th power of the sheaf of ideals $\mathcal{I}^n$.

In [Z], Zariski introduced holomorphic functions along a subvariety. This led to the notion of completing a scheme along a subscheme:

Definition 2.17: Let X be a Noetherian scheme, and let Y be a closed subscheme, defined by a sheaf of ideals $\mathcal{I}$. The **formal completion of X along Y**, denoted $(\hat{X}, \mathcal{O}_{\hat{X}})$, consists of the topological space Y with the sheaf of rings $\mathcal{O}_{\hat{X}} = \varprojlim \mathcal{O}_X/\mathcal{I}^n$. We consider each $\mathcal{O}_X/\mathcal{I}^n$ as a sheaf of rings on Y and make them into an inverse system in the natural way [Ha, II.9, p. 194].

Definition 2.18: Let X, Y and $\mathcal{I}$ be as in the previous definition. Let $\mathcal{F}$ be a coherent sheaf on X. The **completion of $\mathcal{F}$ along** Y, denoted $\hat{\mathcal{F}}$, is the sheaf $\varprojlim \mathcal{F}/\mathcal{I}^n\mathcal{F}$ on Y. It has a natural structure of an $\mathcal{O}_{\hat{X}}$-module [Ha, II.9, p. 194].

Completing schemes along subschemes enables us to define a formal scheme:

Definition 2.19: A **Noetherian formal scheme** is a locally ringed space [Ha, Def. p. 72] $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ which has a finite open cover $\{\mathfrak{U}_i\}$ such that for each i the pair $(\mathfrak{U}_i, \mathcal{O}_{\mathfrak{X}}|_{\mathfrak{U}_i})$ is isomorphic, as a locally ringed space, to the completion of some Noetherian scheme X_i along a closed subscheme (Definition 2.17) [Ha, II.9 p. 194]. If $\mathfrak{X}$ is the completion of a Noetherian scheme X along a subscheme Y, then a sheaf of ideals (resp. modules or algebras) on $\mathfrak{X}$ is called a **formal sheaf of ideals (resp. modules or algebras)** on X.

Definition 2.20: Let $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ be a Noetherian formal scheme (Definition 2.19). A sheaf of ideals $\mathcal{J} \subseteq \mathcal{O}_{\mathfrak{X}}$ is called an **ideal of definition** for $\mathfrak{X}$ if Supp $\mathcal{O}_{\mathfrak{X}}/\mathcal{J} = \mathfrak{X}$ and the locally ringed space $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{J})$ is a Noetherian scheme [Ha II.9 p. 194].

Mock covers, defined above, are one of the tools used in the formal patching process. Another of these tools, **induced covers**, is used in Proposition 3.12 and is defined below.

Definition/Construction 2.21: Let $Z \to Y$ be an H-cover (Definition 2.3) and let H be a subgroup of G. The **induced G-cover** $X = \mathrm{Ind}_H^G Z \to Y$ is a (disconnected) G-cover, consisting of a disjoint union of $[G : H]$ copies of $Z \to Y$. It is constructed by choosing a base point on each copy of Z (over some base point on Y that is not in the branch locus) and choosing a corresponding set of left coset representatives for H in G (one of them being the identity element of G). Thus the connected components of X receive a labeling by the cosets gH, in such a way that $g \in G$ takes the "identity component" (i.e. the connected component labelled by $1_G H$) of X to the component labelled by gH [H3, p. 184]. If, in addition, the cover $Z \to Y$ is a mock cover, the sheets (i.e. irreducible components—Definition 2.1) of Z are labeled by the elements of H. In this case $X = \mathrm{Ind}_H^G Z \to Y$ is also a mock cover whose sheets receive a labeling by the elements of G in such a way that $g \in G$ takes the "identity sheet" (i.e. the sheet labeled by 1_G) of X to the sheet labeled by g.

3. Covers of the line over complete domains

To prove the general result, we first show that for any finite group G, there is a regular irreducible G-Galois cover (Definitions 2.10, 2.12) of $\mathbb{P}^1_{\mathbb{Z}[[x]]}$ and G is regular over $\mathrm{Frac}(\hat{R})$ with a rational point. We use arguments similar to Lemma 2.1–Theorem 2.3 from [H3] that employ mock covers (Definition 2.3) to construct the desired covers. As in Harbater's paper, the technique for building regular irreducible G-Galois covers involves using mock covers and Grothendieck's Existence Theorem [Gr2, Cor. 5.1.6] to patch together cyclic p^n-Galois covers.

We first state a result about rational points from [Ja, p. 265].

LEMMA 3.1: *Let $K(t) \subset L$ be a finite Galois extension of fields with Galois group G such that the corresponding cover (Definition 2.4) of curves has a simple K-rational point. Then K is algebraically closed in L; i.e. G is regular over K.*

Proof: By [JaR, Cor. A2] there exists a place ϕ: $L \to K \cup \{\infty\}$ over K. It follows from [Ja, Lemma 1.2] that K is algebraically closed in L and G is regular over K. ∎

We now follow an argument as in [H3, Lem. 2.1] to show that we can construct irreducible cyclic p^n-Galois covers of $\mathbb{P}^1_{\mathbb{Z}[[x]]}$ and we show these covers mock on the central fibre (Notation 2.6). The construction will allow control over the branch locus (Definition 2.3), which is a necessary condition to later patch the covers together.

We will need to use the following results about the existence of unramified rational points.

PROPOSITION 3.2: *Let $\hat{R}$ be a domain that is complete with respect to a non-zero prime ideal $\mathfrak{a}$. Let $\hat{R}[t] \subset S$ be an extension of domains corresponding to a cover $\operatorname{Spec}(S) \to \mathbb{A}^1_{\hat{R}}$, whose central fibre is a mock cover. Let y be an $\hat{R}$-point of $\mathbb{A}^1_{\hat{R}}$ not meeting the branch locus. Let $K = \operatorname{Frac}(\hat{R})$. Then $\operatorname{Spec}(S)$ contains an unramified $\hat{R}$-point; the generic fibre of the cover, corresponding to the extension of fraction fields, contains an unramified K-point; and $\operatorname{Spec}(S) \to \mathbb{A}^1_{\hat{R}}$ is a regular cover.*

Proof: Let $y_{\mathfrak{a}}$ be the reduction of y mod $\mathfrak{a}$ (i.e. the $\hat{R}/\mathfrak{a}$-point where y meets $\mathbb{A}^1_{\hat{R}/\mathfrak{a}}$). The fibre over this $y_{\mathfrak{a}}$ consists of $\hat{R}/\mathfrak{a}$-points, since $\operatorname{Spec}(S/\mathfrak{a}S) \to \mathbb{A}^1_{\hat{R}/\mathfrak{a}}$ is a mock cover and $y_{\mathfrak{a}}$ does not meet the branch locus. By Hensel's Lemma [B, III.4.3, Thm. 1], the fibre over y consists of $\hat{R}$ points of $\operatorname{Spec}(S)$. The generic fibre of the fibre over y thus consists of $\hat{K}$-points of $\operatorname{Spec}(\operatorname{Frac}(S))$ and these are unramified. It follows from Lemma 3.1 that $\operatorname{Spec}(S) \to \mathbb{A}^1_{\hat{R}}$ is a regular cover. ∎

To construct 2-cyclic covers that meet the above conditions, we may proceed directly. The remaining cases will require some additional lemmas and use results of Saltman [Sa].

LEMMA 3.3: *Let K be a field of characteristic 0, let t be an indeterminate, and let $m \geq 2$ be an integer. Let $F(t) \in K[t]$ be a polynomial of degree ≥ 1 and let $a \in K^\times$. Then*

(a) *$F(t)^m - a$ is not an m-th power in $K[t]$,*

(b) *$Y^m - (F(t)^m - a)$ is irreducible over $K(t)$.*

Proof: (a) Without loss of generality K contains a primitive m-th root of unity ζ_m.

Suppose, by contradiction, that $F(t)^m - a = G(t)^m$, where $G(t) \in K[t]$. Then

$$a = F(t)^m - G(t)^m = \prod_{j=0}^{m-1} \big(F(t) - \zeta_m^j G(t)\big).$$

Hence each factor in the product is of degree 0. In particular,

$$\zeta_m\big(F(t) - G(t)\big) - \big(F(t) - \zeta_m G(t)\big) = (\zeta_m - 1)F(t)$$

is of degree ≤ 0, a contradiction to $\deg F(t) > 0$.

(b) Without loss of generality $(-4)^{1/4} \in K$, so that $-4K(t)^4 = K(t)^4$. Let $r \geq 2$ be an integer that divides m. By (a), with r instead of m, $F(t)^m - a = (F(t)^{m/r})^r - a \notin K(t)^r$. Therefore the assertion follows from [L, Theorem VI.9.1]. ∎

LEMMA 3.4: *Let $f(t) \in \mathbb{Z}[t]$ be a monic polynomial of degree $d \geq 1$ with $f(0) \neq 0$. Then there is an irreducible* 2*-cyclic Galois cover $X \to \mathbb{P}^1_{\mathbb{Z}[x]}$ whose fibre over x, $X_{(x)} \to \mathbb{P}^1_{\mathbb{Z}}$ is a connected mock cover ramified only at (f) (where (f) is regarded as a closed subset of* $\mathrm{Spec}(\mathbb{Z}[t]) = \mathbb{A}^1_{\mathbb{Z}} \subset \mathbb{P}^1_{\mathbb{Z}}$*) and whose pullback to $\hat{X} \to \mathbb{P}^1_{\mathbb{Z}[[x]]}$ is a regular and irreducible cover.*

Proof: Consider $f(t)$ as a polynomial in $\mathbb{Z}[x,t]$. Let $b(x,t) = f(t)^2 - 4x$. It follows from Lemma 3.3 that $b(x,t)$ is not a square in $\mathbb{Z}[x]$ or $\mathbb{Z}[[x]][t]$, the completion of $\mathbb{Z}[x]$ at (x). Since b is the discriminant of the polynomial $y^2 - f(t)y + x$, this polynomial is irreducible over $K = \mathrm{Frac}(\mathbb{Z}[x])$ and $\hat{K} = \mathrm{Frac}(\mathbb{Z}[[x]])$); thus $L = K(t)[y]/(y^2 - f(t)y + x)$ is a field. Let S be the integral closure of $\mathbb{Z}[x,t]$ in L. Then S is a domain and S is finite over $\mathbb{Z}[x,t]$ since $\mathbb{Z}[x,t]$ is a Noetherian normal domain [Mat, Chap. 12, Prop. 31.B]. Now $\mathrm{Spec}(S) \to \mathbb{A}^1_{\mathbb{Z}[x]}$ is an irreducible (Definition 2.7) non-trivial two-to-one cover of $\mathrm{Spec}(\mathbb{Z}[x,t])$ ramified only at $b(x,t)$. The fibre over (x), given by the equation $y^2 - f(t)y = 0$, is a connected mock cover ramified at (f). Consider the cover $X \to \mathbb{P}^1{}_{\mathbb{Z}[x]}$ where X is the normalization of $\mathbb{P}^1_{\mathbb{Z}[x]}$ in $\mathrm{Spec}(S)$. Observe that the fibre over (x) is unramified at the point $t = \infty$. The pullback $\hat{X} \to \mathbb{P}^1_{\mathbb{Z}[[x]]}$ is irreducible since $y^2 - f(t)y + x$ is irreducible in $\hat{K}(t)[y]$. The central fibre $\hat{X}_x \to \mathbb{P}^1_{\mathbb{Z}}$ is mock since it is so generically and is unramified at $t = 0$ since $f(0) \neq 0$. Thus any point of $\mathrm{Spec}(\mathbb{Z}[[x]][t])$ whose reduction mod (x) is (t) is unramified. It follows from Proposition 3.2 that the cover is regular. ∎

To construct the remaining covers—i.e. odd prime power cyclic covers and the remaining even cyclic covers—we will work in steps. We first construct cyclic covers of $\mathbb{A}^1_{\mathbb{Z}[x]}$ that are mock on the fibre over (x). We then show we can control the branch locus on this fibre. Finally we complete these to covers of $\mathbb{P}^1_{\mathbb{Z}[x]}$, show that the fibre over $x = 0$ is connected and unramified at infinity and that the pullbacks to $\mathbb{P}^1_{\mathbb{Z}[[x]]}$ are regular and irreducible.

We introduce some notation to be used throughout this section. Let $K = \mathrm{Frac}(\mathbb{Z}[x]) = \mathbb{Q}(x)$. Let p be a prime number, let $q = p^n$ for some n and let ζ_q be a primitive qth root of unity. Define $K' = K[\zeta_q]$ and let R' be the integral closure of $\mathbb{Z}[x]$ in K' (i.e. $R' = \mathbb{Z}[\zeta_q, x]$).

Now we are ready to show the existence of odd irreducible p^n-cyclic Galois covers of $\mathbb{A}^1_{\mathbb{Z}[x]}$ that are mock on the fibre over (x). We state the following proposition that we will prove by cases in subsequent lemmas. The lemmas follow the proofs of the cases and subcases in [H2, Prop. 2.2,ii].

PROPOSITION 3.5: *Let $f(t) \in \mathbb{Z}[t]$ be a monic polynomial of degree $d \geq 1$ with $f(0) \neq 0$. Let $q \neq 2$ be a prime power. Then there is an irreducible q-cyclic Galois cover $\mathrm{Spec}(S) \to \mathbb{A}^1_{\mathbb{Z}[x]}$ whose fibre over (x), $\mathrm{Spec}(S/xS) \to \mathbb{A}^1_{\mathbb{Z}}$, is a mock cover. Moreover, this cover may be chosen so that the pullback to the cover $\mathrm{Spec}(S') \to \mathbb{A}^1_{R'}$ has a corresponding extension of fraction fields $K'(t) \subset \mathrm{Frac}(S')$ given by a polynomial of the form $y^q - B(x,t)$, where $B(x,t) \in \mathbb{Z}[x,t]$, $B(x,t) \equiv f(t)^r \bmod (x)$ for some $r \geq 1$ and $y^q - B(x,t)$ is irreducible over $\mathbb{Z}[[x]][\zeta_q, t]$.*

We first prove Proposition 3.5 for odd prime powers, following [H2, Prop. 2.2, Case I].

LEMMA 3.6: *Let $f(t) \in \mathbb{Z}[t]$ be a monic polynomial of degree $d \geq 1$ with $f(0) \neq 0$. Let q be an odd prime power. Then Proposition 3.5 is true for q.*

Proof: Consider $f(t)$ as an element of $\mathbb{Z}[x,t]$. Write $q = p^n$ for some odd prime number p.

We will use Saltman [Sa, Thm. 2.3] to construct a q-cyclic extension of $\mathbb{Z}[x,t]$.

Since q is odd, $\mathrm{Gal}(K'/K)$ is a cyclic group of order $s < q$, with generator $\tau\colon \zeta_q \to \zeta_q^m$, where m is a generator of $(\mathbb{Z}/q\mathbb{Z})^\times$. Note that $m^s \equiv 1 \pmod q$ so there is some k such that $m^s - qk = 1$.

Let $b(x,t) = f(t)^q - \zeta_q p^2 x$. It follows from Lemma 3.3 that $b(x,t)$ is not a p-th power in $R'[t]$ or in $\mathbb{Z}[[x]][\zeta_q, t]$.

Let $B(x,t) = M(b)$ where (as in [Sa, Lemma 2.2])

$$M(b) = \prod_{i=0}^{s-1} \tau^i(b)^{m^{s-1-i}} = \prod_{i=0}^{s-1} \left(f(t)^q - \tau^i(\zeta_q)p^2 x\right)^{m^{s-1-i}}.$$

Observe that $B(x,t)$ is a product of s relatively prime polynomials in in $E[t]$, where E is some field that contains $\mathbb{Q}(\zeta_q, x)$. By Lemma 3.3 none of the factors is a p-th power of a polynomial in $E[t]$, thus $B(x,t)$ is not a p-th power.

By [L, Theorem VI.9.1], $y^q - B(x,t)$ is irreducible over $R'[t]$ and $\mathbb{Z}[[x]][\zeta_q, t]$, thus $L' = K'(t)[y]/(y^q - B(x,t))$ is a field with $\mathrm{Gal}(L'/K'(t))$ generated by $\sigma : y \to \zeta_q y$. As in [H2, Prop. 2.2], since

$$(y^m b^{-k})^{p^n} = M(b)^m b^{-kp^n} = \tau(M(b)) = \tau(y^{p^n}),$$

τ can be extended to an automorphism of L' over $K(t)$ by defining $\tau(y) := y^m b(x,t)^{-k}$ where k, as defined above, is the integer such that $m^s - kq = 1$. The order of τ is s, since

$$\tau^s(y) = y^{m^s} M(b)^{-k} = y.$$

By [Sa, Thm. 2.3], σ and τ commute and L' descends to a q-cyclic Galois extension of $K(t)$, i.e. there exists an extension L of $K(t)$ such that

$$L' = L \otimes_K K'(t)$$

where L is the fixed field of $\tau : L' \to L'$. Let S', S be the integral closures of $\mathbb{Z}[x,t]$ in L', L. Now S and S' are domains and are finite over $\mathbb{Z}[x,t]$ since $\mathbb{Z}[x,t]$ is Noetherian and normal [Mat, Chap. 12, Prop. 31.B]. Therefore $\mathrm{Spec}(S) \to \mathbb{A}^1_{\mathbb{Z}[x]}$ is an irreducible cover of $\mathbb{A}^1_{\mathbb{Z}[x]}$ It follows from Remark 2.8.b that the cover is locally irreducible. The pullback $\mathrm{Spec}(S') \to \mathbb{A}^1_{R'}$ has a corresponding extension of fraction fields $K'(t) \subseteq \mathrm{Frac}(S') = L'$. The extension $K'(t) \subset L'$ is given by the polynomial $y^q - B(x,t)$ and $B(x,t) \equiv f(t)^r \mathrm{mod}(x)$ for appropriate r.

We now show the fibre over the ideal (x), $\mathrm{Spec}(S/xS) \to \mathbb{A}^1_{\mathbb{Z}}$, is a mock cover.

Let $z = y + \tau(y) + \tau^2(y) + \cdots + \tau^{s-1}(y) \in S'$. We defined L as the fixed field of τ.

CLAIM: $L = K(t)[z]$.

To prove the claim, observe that $z \in L$ and $[L : K(t)] = [L' : K'(t)] = q$, so it suffices to show that $[K(t)[z] : K(t)] \geq q$.

Consider the basis $1, y, \ldots, y^{q-1}$ of L' over $K'(t)$. Since $y^q = B(x,t) \in K'(t)^\times$, for each integer j the element y^j lies in the $K'(t)$-subspace spanned by $y^{\bar{j}}$, where $\bar{j} = j \bmod q$. So if $j_1, \ldots, j_s$ are integers non-congruent modulo q, then $y^{j_1}, \ldots, y^{j_s}$ are linearly independent over $K'(t)$. In particular, since s is the order of m in $(\mathbb{Z}/q\mathbb{Z})^\times$, the numbers $1, m, \ldots, m^{s-1}$ are non-congruent modulo q, and hence $y, y^m, \ldots, y^{m^{s-1}}$ are linearly independent over $K'(t)$. But from $\tau(y) := y^m b(x,t)^{-k}$ we see that $y^{m^i} \in \tau^i(y) K'(t)$, and so $y, \tau(y), \ldots, \tau^{s-1}(y)$ are linearly independent over $K'(t)$.

Since $\sigma(y) = \zeta_q y$,

$$\begin{aligned}\sigma^i(z) &= \sigma^i(y) + \tau(\sigma^i(y)) + \cdots + \tau^{s-1}(\sigma^i(y)) \\ &= \zeta_q^i y + \tau(\zeta_q^i)\tau(y) + \cdots + \tau^{s-1}(\zeta_q^i)\tau^{s-1}(y)\end{aligned}$$

and hence $\sigma^i(z) = \sigma^j(z)$ iff $\zeta_q^i = \zeta_q^j$ iff $i \equiv j \pmod q$. Thus z has q distinct conjugates over $K'(t)$, and hence $[K(t)[z] : K] \geq [K'(t)[z] : K'] \geq q$.

We follow an argument from [H2, Prop 2.2] to construct a section of $\mathrm{Spec}(S) \to \mathbb{A}^1_{\mathbb{Z}[x]}$ over $\mathbb{A}^1_{\mathbb{Z}}$. Define $S_0' = R'[t,y]/(y^q - B(x,t))$ and observe that S' is the integral closure of S_0' since $B(x,t) \in R'[t]$. Recall that $\tau^i(b) \equiv f(t)^q \bmod x$. Define a homomorphism $\phi_0' : S_0' \to (R'/xR')[t]$ by sending each element of $R'[t]$ to its canonical image in $(R'/xR')[t] = \mathbb{Z}[\zeta_q, t]$ (note that $x \mapsto 0$) and sending $y \mapsto f^{1+m+m^2+\dots+m^{s-1}}$ (or in Saltman's notation, $y \mapsto M(f)$). We would like to extend ϕ_0' to a map from S' into $(R'/xR')[t]$. Consider the point $a = (0, M(f))$ on the K' curve $y^q - B(0,t)$. The point a is K'-rational, and a is simple, since we chose f so that $f(0) \neq 0$ and hence the partial derivative $\partial/\partial y$ of the curve is not zero at this point. It follows from [JaR, Cor. A3] that ϕ_0' can be extended to a map $L' \to \mathrm{Frac}((R'/xR')[t])$. Since S' is the integral closure of S_0' and $(R'/xR')[t]$ is integrally closed we conclude that ϕ_0' extends to a homomorphism ϕ': $S' \to (R'/xR')[t]$.

Now

$$\phi'\tau(y) = \phi'(y^m b^{-k}) = f^{m+m^2+m^3+\dots+m^s-kq} = f^{1+m+m^2+\dots+m^{s-1}} = \phi'(y)$$

since $m^s - kq = 1$. Thus $\phi'\tau = \phi'$.

Observe that z is integral over $\mathbb{Z}[x,t]$ since $\tau(y) = y^m b^{-k}$ is a root of the monic polynomial $Z^q - \tau(B(x,t))$. Hence S is the integral closure of $S_0 = \mathbb{Z}[x,t,z]$ in L. Consider $\phi = \phi' \mid_S$ and note $\phi(z) = \phi(y + \tau(y) + \dots + \tau^{s-1}(y)) = sf^{1+m+m^2+\dots+m^{s-1}} \in \mathbb{Z}[t]$. Now we have ϕ: $S \to \mathbb{Z}[t]$ and this induces $S/xS \to \mathbb{Z}[t]$. Thus $\mathrm{Spec}(S) \to \mathbb{A}^1_{\mathbb{Z}[x]}$ has a section over $\mathbb{A}^1_{\mathbb{Z}}$. But this cover is Galois, so we have shown the fibre over (x) is a mock cover. ∎

We continue by proving Proposition 3.5 for powers of 2, i.e. we show the existence of 2^n-cyclic covers ($n > 1$) of $\mathbb{A}^1_{\mathbb{Z}[x]}$ with mock fibres over (x). We treat the case $n = 2$ separately, following [H2, Prop. 2.2, Case II, Subcase b].

LEMMA 3.7: *Proposition 3.5 is true for $q = 4$.*

Proof: First note that $K' = K(\zeta_4) = K(i)$. Let κ: $i \mapsto -i$ (complex conjugation) be the generator of the Galois group $\mathrm{Gal}(K'/K)$. Consider $f(t)$ as an element of $\mathbb{Z}[x,t]$. Let $B(x,t) = (f(t) + 4ix)^3(f(t) - 4ix)$ and define

$$L' = K'(t)[y]/(y^4 - B(x,t)).$$

By Lemma 3.3, $y^q - B(x,t)$ is irreducible in $R'[x,t]$ and $\mathbb{Z}[[x]][\zeta_q t]$. Hence L' is a field. The extension is a 4-cyclic Galois extension with Galois group generated by σ: $y \mapsto iy$. Since

$$(y^{-1}g)^4 = b^{-3}\kappa(b)^{-1}g^4 = \kappa(b^3\kappa(b)) = \kappa(y^4),$$

we may extend κ to an automorphism of L' over $K(t)$ by defining $\kappa(y) = y^{-1}g$ where $g = (f^2 + 16x^2)$. From the proof of [Sa, Thm. 2.4] (taking in Saltman's notation, $b = f - 4ix, z = f^2 + 16x^2 = g, a = B(x,t)$), κ is well-defined, $\sigma\kappa = \kappa\sigma$ and L' descends to a 4-cyclic extension L of $K(t)$ where L is the fixed field of κ.

Let S', S be the integral closures of $\mathbb{Z}[x,t]$ in L', L. Then S and S' are domains and are finite over $\mathbb{Z}[x,t]$ since $\mathbb{Z}[x,t]$ is a Noetherian normal domain [Mat, Chap. 12, Prop. 31.B]. Therefore $\operatorname{Spec}(S) \to \mathbb{A}^1_{\mathbb{Z}[x]}$ is an irreducible cover. It follows from Remark 2.8.b that the cover is locally irreducible. By the definition of S', the pullback $\operatorname{Spec}(S') \to \mathbb{A}^1_{R'}$ has a corresponding extension of fraction fields $K'(t) \subseteq \operatorname{Frac}(S') = L'$. The extension $K'(t) \subset L'$ is given by the polynomial $y^q - B(x,t)$ and $B(x,t) \equiv f(t)^r \operatorname{mod}(x)$ for appropriate r.

We now show the fibre over (x) is a mock cover. Let $z = y + \kappa(y) \in S'$. We defined L as the fixed field of κ. Note that $L = K(t)[z]$ since the $\sigma^j(z) = i^j y + (-i)^j \kappa(y)$ are distinct for $0 \leq j \leq 3$.

We follow an argument from [H2, Prop. 2.2] to construct a section of $\operatorname{Spec}(S) \to \mathbb{A}^1_{\mathbb{Z}[x]}$ over $\mathbb{A}^1_{\mathbb{Z}}$. Define $S'_0 = R'[t,y]/(y^q - B(x,t)) = \mathbb{Z}[i,x,t,y]$ and observe that S' is the integral closure of S'_0 since $B(x,t) \in R'[t]$. Define a homomorphism ϕ'_0: $S'_0 \to (R'/xR')[t]$ by sending each element of $R'[t]$ to its canonical image in $(R'/xR')[t]$ and sending $y \mapsto f$. Note that $\phi'_0(x) = 0$. This homomorphism extends to a homomorphism ϕ': $S' \to (R'/xR')[t]$ since S' is the integral closure of S'_0 and $(R'/xR'[t])$ is integrally closed. Observe that $z \in S'$ since $\kappa(y) = y^{-1}g$ satisfies the monic polynomial $Z^4 - B(x,t)$. Thus S is the integral closure of $S_0 = \mathbb{Z}[x,t,z]$ in L. Consider $\phi = \phi' \mid_S$ and note $\phi(z) = \phi(y + \kappa(y)) = 2f \in \mathbb{Z}[x,t]$. Therefore ϕ: $S \to \mathbb{Z}[t]$ and so $\operatorname{Spec}(S) \to \mathbb{A}^1_{\mathbb{Z}[x]}$ has a section over $\mathbb{A}^1_{\mathbb{Z}}$. But this cover is Galois, so the fibre over (x) is a mock cover. ∎

We finish the cases needed to prove Proposition 3.5 by constructing the desired 2^n-cyclic Galois covers for $n > 2$, following [H2, Prop. 2.2, Case II, Subcase c].

LEMMA 3.8: *Let $f(t) \in \mathbb{Z}[t]$ be a monic polynomial of degree $d \geq 1$ with $f(0) \neq 0$. Let $n > 2$ and $q = 2^n$. Then Proposition 3.5 is true for q.*

Proof: If $n > 2$, then $K' = K(\zeta_q)$ is Galois over K with Galois group

$$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/s\mathbb{Z}$$

where $s \in 2\mathbb{Z}$ and $s = 2^{n-2}$, with the first factor generated by κ: $\zeta_q \to \zeta_q^{-1}$ (complex conjugation) and the second factor generated by τ: $\zeta_q \to \zeta_q^5$.

Consider $f(t)$ as an element of $\mathbb{Z}[x,t]$. Let $b(x,t) = f(t)^q + 4\zeta_q x$ and let $a = b^{2^{n-1}+1}\kappa(b)^{2^{n-1}-1}$. Now define $B(x,t) = M(a)$ where $M(a)$ is Saltman's construction, described in Lemma 3.6 (with $m = 5$), and let $k = (5^s - 1)/q$. By Lemma 3.3, $b(x,t)$ is not a square and hence a and $B(x,t)$ are not squares in $R'[t]$ or $\mathbb{Z}[[x]][\zeta_q, t]$. Thus $y^q - B(x,t)$ is irreducible in $R'[t]$ and $\mathbb{Z}[[x]][\zeta_q, t]$. Let $L' = K'(t)[y]/(y^q - B(x,t))$ with $\mathrm{Gal}(L'/K'(t))$ generated by $\sigma\colon y \to \zeta_q y$. Then L' is a field. We may extend κ and τ to automorphism of L' over $K'(t)$ as follows: Since

$$(y^5 a^{-k})^{2^n} = M(a)^5 a^{-2^n k} = \tau(M(a))$$

we may define $\tau(y) = y^5 a^{-k}$. Since κ and τ commute, we have

$$M(a)\kappa(M(a)) = M(a)M(\kappa(a)) = M(a\kappa(a)) = M(b^{2^n}\kappa(b)^{2^n}),$$

so we may define $\kappa(y) = y^{-1}M(b\kappa(b))$. One may check that the orders of κ and τ are preserved ($\mathrm{ord}(\kappa) = 2, \mathrm{ord}(\tau) = s$) and that κ, τ and σ are pairwise commutative. By [Sa, Thm. 2.7], L' descends to a q-cyclic extension, L of $K(t)$ where L is the fixed field of the subgroup of $\mathrm{Gal}(L'/K(t))$ generated by κ and τ.

Let S, S' be the integral closures of $\mathbb{Z}[x,t]$ in L, L'. Then S and S' are domains and are finite over $\mathbb{Z}[x,t]$ since $\mathbb{Z}[x,t]$ is a Noetherian normal domain [Mat, Chap. 12, Prop. 31.B]. Therefore $\mathrm{Spec}(S) \to \mathbb{A}^1_{\mathbb{Z}[x]}$ is an irreducible cover. It follows from Remark 2.8.b that the cover is locally irreducible. The pullback $\mathrm{Spec}(S') \to \mathbb{A}^1_{R'}$ has a corresponding extension of fraction fields $K'(t) \subseteq \mathrm{Frac}(S') = L'$. The extension $K'(t) \subset L'$ is given by the polynomial $y^q - B(x,t)$ and $B(x,t) \equiv f(t)^r \mathrm{mod}(x)$ for appropriate r.

We now proceed to show the fibre over (x) is a mock cover.

Let $z = \sum_{i=1}^{s-1}(\tau^i(y) + \tau^i\kappa(y)) \in S'$. We defined L as the fixed field of τ and κ.

Claim: $L = K(t)[z]$.

To prove the claim, it suffices to show, as in Lemma 3.6, that z has q distinct conjugates over $K'(t)$. As there, if $\{j_i\}_i$ are integers non-congruent modulo q, then $\{y^{j_i}\}_i$ are linearly independent over $K'(t)$. In particular,

$$y, y^5, \ldots, y^{5^{s-1}}, y^{-1}, y^{-5}, \ldots, y^{-5^{s-1}}$$

are linearly independent over $K'(t)$. By the definitions of $\kappa(y)$ and $\tau(y)$, so are

$$y, \tau(y), \ldots, \tau^{s-1}(y), \kappa(y), \kappa\tau(y), \ldots, \kappa\tau^{s-1}(y).$$

Since

$$\sigma^j(z) = \sum_{i=1}^{s-1}(\zeta_q^{j5^i}\tau^i(y) + \zeta_q^{-j5^i}\tau^i\kappa(y)),$$

$\sigma^j(z) = \sigma^{j'}(z)$ iff $\zeta_q^j = \zeta_q^{j'}$ iff $\zeta_q^{j5^i} = \zeta_q^{j'5^i}$ for every i iff $j \equiv j' \pmod q$.

We again follow [H2, Prop 2.2] to construct a section of $\mathrm{Spec}(S) \to \mathbb{A}^1_{\mathbb{Z}[x]}$ over $\mathbb{A}^1_{\mathbb{Z}}$. Once again define $S_0' = R'[t,y]/(y^q - B(x,t))$ and oberve S' is the integral closure of S_0' since $B(x,t) \in R'[t]$. Define a homomorphism ϕ_0': $S_0' \to (R'/xR')[t]$ by sending each element of $R'[t]$ to its canonical image in $(R'/xR')[t]$ and sending $y \mapsto f^{q(1+5+5^2+\cdots+5^{s-1})}$ (or in Saltman's notation, $y \mapsto M(f^q)$). This extends to a homomorphism ϕ': $S' \to (R'/xR')[t]$ since S' is the integral closure of S_0' and $(R'/xR')[t]$ is integrally closed. Now

$$\phi'\tau(y) = \phi'(y^5a^{-k}) = f^{q(5+5^2+5^3+\cdots+5^s)-kq^2} = f^{q(1+5+5^2+\cdots+5^{s-1})} = \phi'(y)$$

since $5^s - kq = 1$. Thus $\phi'\tau = \phi'$. Also $\phi'(\kappa(y)) = M(f) = \phi'(y)$.

Observe that z is integral over $\mathbb{Z}[x,t]$ since $\tau(y) = y^5a^{-k}$ is a root of the monic polynomial $Z^{p^n} - B(x,t)^5a^{-qk}$ and $\kappa(y) = y^{-1}M(b/\kappa(b))$ satisfies the monic polynomial $Z^q - \kappa(B(x,t))$. Thus S is the integral closure of $S_0 = \mathbb{Z}[x,t,z]$ in L. Consider $\phi' |_S$ and note $\phi(z) = \phi(\sum_{i=1}^{s-1}\tau^i(y) + \tau^i\kappa(y)) = 2s(f^{1+5+5^2+\cdots+5^{s-1}}) \in \mathbb{Z}[t]$. Therefore ϕ: $S \to \mathbb{Z}[t]$ and so $\mathrm{Spec}(S) \to \mathbb{A}^1_{\mathbb{Z}[x]}$ has a section over $\mathbb{A}^1_{\mathbb{Z}}$. But this cover is Galois, so the fibre over (x) is a mock cover. ∎

We may now prove Proposition 3.5:

Proof: The result follows immediately from Lemma 3.6 if q is odd, Lemma 3.7 if $q = 4$ and Lemma 3.8 if $q = 2^n$ with $n > 2$. ∎

We have constructed irreducible q-cyclic Galois covers $\mathrm{Spec}(S) \to \mathrm{Spec}(\mathbb{Z}[x,t])$ that are mock on the fibre over (x). We continue by showing that these mock fibres are ramified only over the polynomial (f) that was specified in Proposition 3.5.

LEMMA 3.9: *Let $f(t) \in \mathbb{Z}[t]$ be a monic polynomial of degree $d \geq 1$ with $f(0) \neq 0$ and let $q = p^n$ be a prime power. Then there is an irreducible q-cyclic Galois cover $\mathrm{Spec}(S) \to \mathbb{A}^1_{\mathbb{Z}[x]}$ whose fibre over (x), $\mathrm{Spec}(S/xS) \to \mathbb{A}^1_{\mathbb{Z}}$, is a connected mock cover ramified only at (f) (where (f) is regarded as a closed subset of $\mathrm{Spec}(\mathbb{Z}[t]) = \mathbb{A}^1_{\mathbb{Z}}$). Moreover, this cover may be chosen so that the pullback to the cover $\mathrm{Spec}(S') \to \mathbb{A}^1_{R'}$ has a corresponding extension of fraction fields $K'(t) \subset \mathrm{Frac}(S')$ given by a polynomial of the form $y^q - B(x,t)$ where $B(x,t) \in \mathbb{Z}[x,t]$ and $B(x,t) \equiv f(t)^r \bmod (x)$ for some $r \geq 1$ and $y^q - B(x,t)$ is irreducible over $\mathbb{Z}[[x]][\zeta_q, t]$.*

Proof: If $q = 2$ the conclusion follows from Lemma 3.4.

If $q \neq 2$, there exist irreducible q-cyclic Galois covers $\mathrm{Spec}(S) \to \mathbb{A}^1_{\mathbb{Z}[x]}$ that are mock on the fibre over (x), by Proposition 3.5. As in the proposition, we may assume that the pullback to the cover $\mathrm{Spec}(S') \to \mathbb{A}^1_{R'}$ has a corresponding extension of fraction fields $K'(t) \subset \mathrm{Frac}(S')$ given by a polynomial of the form $y^q - B(x,t)$, where $B(x,t) \in \mathbb{Z}[x,t]$, $B(x,t) \equiv f(t)^r \bmod (x)$ for some $r \geq 1$ and $y^q - B(x,t)$ is irreducible over $\mathbb{Z}[[x]][\zeta_q, t]$.

Let $L' = \mathrm{Frac}(S')$, $L = \mathrm{Frac}(S)$.

Let $\mathfrak{I}$ be the ideal in $R'[t]$ generated by $B(x,t)$. The extension $\mathbb{Z}[x,t] \subset S'$ is unramified except over the contraction in $\mathbb{Z}[x,t]$ of $\mathfrak{I}$ and possibly primes containing (p). Thus $\mathbb{Z}[x,t] \subset S$ is unramified except over these primes. We proceed to show that $\mathbb{Z}[x,t]$ is unramified over primes containing (p). We will therefore conclude that $\mathbb{Z}[x,t] \subset S$ is ramified only over the contraction of $\mathfrak{I}$. Thus in the fibre over (x), it is ramified only over (f).

First note the extension $\mathbb{Z}[x,t] \subset R'[t]$ is ramified only at primes containing (p). Now we show $R'[t] \subset S'$ is ramified only over $\mathfrak{I}$:

Let P be a prime ideal of S' containing (p) that does not contain $B(x,t)$. Note that such a P exists since $B(x,t) \equiv f(t)^{rq} \bmod p^2$ and $p \not| f(t)$ since $f(t)$ is monic, hence $B(x,t) \notin \sqrt{(p)}$. Let $\mathfrak{p} = P \cap R'[t]$ and let $\hat{\mathcal{O}}$ be the completion of $R'[t]$ at the prime $\mathfrak{p}$.

Let

$$\gamma = \frac{y}{f(t)^r} \in L'$$

and let

$$a = \gamma^q = \frac{B(x,t)}{f(t)^{qr}}.$$

Since $p^2 \subset \mathfrak{p}$ and $B(x,t) \notin \mathfrak{p}$, we have $f(t) \notin \mathfrak{p}$. Thus $a \in R'[t]_{\mathfrak{p}}$ and $a \equiv 1 \bmod p^2$. Hence $a = 1 + p^2\rho$ for some $\rho \in \hat{\mathcal{O}}$.

Observe that $p^n \nmid n!$, hence $\gamma = a^{1/p} \in \hat{\mathcal{O}}$, since we may write the Taylor expansion of $\gamma = a^{1/p}$ as

$$a^{1/p} = \sum_{n=0}^{\infty} \binom{\frac{1}{p}}{n} (p^2\rho)^n = \sum_{n=0}^{\infty} \frac{\prod_{k=1}^{n}(1-kp)}{n!p^n} p^{2n}\rho^n = \sum_{n=0}^{\infty} \prod_{k=1}^{n} (1-kp)\frac{p^n}{n!}\rho^n.$$

As $L' = K'(t)(y) = K'(t)(\gamma)$, it follows that L' is contained in the quotient field of $\hat{\mathcal{O}}$. But since $\hat{\mathcal{O}}$ is integrally closed and S' is the integral closure of $R'[t]$ in L, we get $S' \subset \hat{\mathcal{O}}$, and, since $\hat{\mathcal{O}}$ is a local ring, $S'_P \subset \hat{\mathcal{O}}$. As $\hat{\mathcal{O}}/R'[t]_{\mathfrak{p}}$ is unramified, so is $S'_P/R'[t]_{\mathfrak{p}}$, i.e., $S'/R'[t]$ is unramified at P and so the extension $R'[t] \subset S'$ is ramified only over $B(x,t)$.

To show that the extension $\mathbb{Z}[x,t] \subset S$ is unramified at primes containing (p) we will again work in the complete local ring.

Let $P \subset \mathbb{Z}[x,t]$ be a prime containing (p) but not the contraction of $B(x,t)$. Then by the Lying Over Theorem, $P = \mathfrak{p} \cap \mathbb{Z}[x,t]$ for some prime $\mathfrak{p} \subset R'[t]$ such that $\mathfrak{p}$ contains (p) but not $(B(x,t))$. Let $\hat{\mathcal{O}}_{\mathbb{Z}[x,t]_P}$ be the completion of $\mathbb{Z}[x,t]_P$. The extension $\mathbb{Z}[x,t] \subset R'[t]$ is totally ramified at P, since the extension $\mathbb{Z} \subset \mathbb{Z}[\zeta_q]$ (where $q = p^n$) is totally ramified at (p). Hence $\hat{\mathcal{O}}_{R'[t]_\mathfrak{p}} = R'[t] \otimes_{\mathbb{Z}[x,t]} \hat{\mathcal{O}}_{\mathbb{Z}[x,t]_P}$ is a domain. By the above $\hat{S}' = S' \otimes_{\mathbb{Z}[x,t]} \hat{\mathcal{O}}_{\mathbb{Z}[x,t]_P} = \hat{\mathcal{O}}_{R'[t]_\mathfrak{p}}$ so $\hat{S}'$ and $\hat{S} = S \otimes_{\mathbb{Z}[x,t]} \hat{\mathcal{O}}_{\mathbb{Z}[x,t]_P}$ are domains. Recall that

$$S \otimes_{\mathbb{Z}[x,t]} R'[t] = S'.$$

Tensoring both sides of this equation with $\hat{\mathcal{O}}_{\mathbb{Z}[x,t]_P}$ yields

$$\hat{S} \otimes_{\hat{\mathcal{O}}_{\mathbb{Z}[x,t]_P}} \hat{\mathcal{O}}_{R'[t]_\mathfrak{p}} = \hat{S}'.$$

The above are domains; thus we have shown that $\hat{S}$ is linearly disjoint from $\hat{\mathcal{O}}_{R'[t]_\mathfrak{p}}$ over $\hat{\mathcal{O}}_{\mathbb{Z}[x,t]_P}$. Since $\hat{S} \subset \hat{S}'$ and $\hat{S}' = \hat{\mathcal{O}}_{R'[t]_\mathfrak{p}}$ we have $\hat{S} \subset \hat{\mathcal{O}}_{R'[t]_\mathfrak{p}}$. These two domains are linearly disjoint over $\hat{\mathcal{O}}_{\mathbb{Z}[x,t]_P}$ so their intersection is $\hat{\mathcal{O}}_{\mathbb{Z}[x,t]_P}$. We conclude $\hat{S} = \hat{\mathcal{O}}_{\mathbb{Z}[x,t]_P}$ so the extension $\hat{\mathcal{O}}_{\mathbb{Z}[x,t]_P} \subset \hat{S}$ is étale. Therefore the extension $\mathbb{Z}[x,t]_P \subset S \otimes_{\mathbb{Z}[x,t]} \mathbb{Z}[x,t]_P$ is étale and thus unramified at P. [Ha, III, Ex. 10.3,4]

We showed earlier that $\mathbb{Z}[x,t] \subset S$ is ramified only at the contraction of $\mathfrak{I}$ and possibly primes containing (p). We have just proven $\mathbb{Z}[x,t] \subset S$ is unramified at primes P that contain (p) but not the contraction of $(B(x,t))$; hence $\mathbb{Z}[x,t] \subset S$ is ramified only over the contraction of $\mathfrak{I}$. So modulo (x) is ramified only at (f).

It remains to show the fibre over (x), $\operatorname{Spec}(S/xS)$, is connected. The extension $\mathbb{Z}[t] \subset S/xS$ is totally ramified over (f). By the Lying Over Theorem, the fibre over any maximal ideal containing (f) is non-empty and, as the extension is totally ramified over (f), it consists of a single point. Hence $\operatorname{Spec}(S/xS)$ is connected. ∎

We have constructed $q = p^n$-cyclic covers of $\mathbb{A}^1_{\mathbb{Z}[x]}$ that are mock on the fibre over (x) and have specified branch loci. To complete these to G-Galois covers $X \to \mathbb{P}^1_{\mathbb{Z}[x]}$ whose fibres over (x) are unramified at the point $t = \infty$ and whose pullbacks to G-Galois covers $\hat{X} \to \mathbb{P}^1_{\mathbb{Z}[[x]]}$ are regular and irreducible, we use the following lemma:

Lemma 3.10: *Let $q = p^n$ for some prime p, let l be a positive integer. Let $C(x,t) \in \mathbb{Z}[x,t]$ be a monic polynomial in t. Let $\bar{C}(t)$ be the reduction of $C(x,t)$*

mod(x) and suppose the degree in t of $C(x,t) = \deg\bar{C}(t) = ql$. Further, assume that $C(x,t)$ is not a p-th power in $\mathbb{Q}(x,t)$. Then $L = \mathbb{Q}(x,t)[y]/(y^q - C(x,t))$ is a field. Let S be the integral closure of $\mathbb{Z}[x,t]$ in L, corresponding to a morphism $\mathrm{Spec}(S) \to \mathbb{A}^1_{\mathbb{Z}[x]}$. Let X be the normalization of $\mathbb{P}^1_{\mathbb{Z}[x]}$ in $\mathrm{Spec}(S)$. Then the fibre of the morphism $X \to \mathbb{P}^1_{\mathbb{Z}[x]}$ over (x), i.e. $X_x \to \mathbb{P}^1_{\mathbb{Z}}$, is unramified over $t = \infty$.

Proof: By [L, Theorem VI.9.1], $y^q - C(x,t)$ is irreducible, hence L is a field.

Let $S_0 = \mathbb{Z}[t,x,y]/y^q - C(x,t)$. Observe that S is the integral closure of S_0 in L. Now $\mathbb{A}^1_{\mathbb{Z}[x]}$ is the affine t-line over $\mathbb{Z}[x]$. Consider this as one affine patch of the projective t-line over $\mathbb{Z}[x]$ where the other affine patch has coordinate $\bar{t}$ and the transition function on the overlap is given by $t\bar{t} = 1$. Consider the morphism $X_0 \to \mathbb{P}^1_{\mathbb{Z}[x]}$ given on the t patch by the above extension of domains and given on the $\bar{t}$ patch by the ring extension $\mathbb{Z}[x,\bar{t}] \subset \bar{S}_0 = \mathbb{Z}[x,\bar{t},\bar{y}]/\bar{y}^q - \widetilde{C}(x,\bar{t})$ where $\bar{y} = \bar{t}^l y$ and $\widetilde{C}(x,\bar{t}) = \bar{t}^{lq}C(x,t)$. Since $\bar{y}^q - \widetilde{C}(x,\bar{t}) = \bar{t}^{lq}(y^q - C(x,t))$, the morphisms $\mathrm{Spec}(S_0) \to \mathrm{Spec}(\mathbb{Z}[x,t])$, $\mathrm{Spec}(\bar{S}_0) \to \mathrm{Spec}(\mathbb{Z}[x,\bar{t}])$ agree on the overlap, $\mathrm{Spec}(\mathbb{Z}[x,t,\bar{t}]/(t\bar{t}-1))$ via the transition function $\bar{y} = \bar{t}^l y$. Since $C(x,t)$ is monic as a polynomial in t, by definition of S_0, $\mathbb{Z}[x,\bar{t}] \subset \bar{S}_0$ is étale over $\bar{t} = 0$, i.e. $X_0 \to \mathbb{P}^1_{\mathbb{Z}[x]}$ is étale over $t = \infty$. Observe that X is the normalization of X_0. The fibre $X_x \to \mathbb{P}^1_{\mathbb{Z}}$ is unramified over $t = \infty$ since the equations defining the fibre have the same t degree as the equations defining the cover. ∎

Now we may prove the following lemma.

LEMMA 3.11: *Let $f(t) \in \mathbb{Z}[t]$ be a monic polynomial of degree $d \geq 1$ with $f(0) \neq 0$ and let q be a prime power. Then there is an irreducible q-cyclic Galois cover $\mathrm{Spec}(S) \to \mathbb{A}^1_{\mathbb{Z}[x]}$ whose fibre over (x), $\mathrm{Spec}(S/xS) \to \mathbb{A}^1_{\mathbb{Z}}$, is a connected mock cover ramified only at the locus of (f) in $\mathrm{Spec}(\mathbb{Z}[t]) = \mathbb{A}^1_{\mathbb{Z}} \subset \mathbb{P}^1_{\mathbb{Z}}$ and whose pullback to $\hat{X} \to \mathbb{P}^1_{\mathbb{Z}[[x]]}$ is a regular irreducible cover. Moreover, this cover may be chosen so that the pullback to the cover $\mathrm{Spec}(S') \to \mathbb{A}^1_{R'}$ has a corresponding extension of fraction fields $K'(t) \subset \mathrm{Frac}(S')$ given by a polynomial of the form $y^q - B(x,t)$ where $B(x,t) \in \mathbb{Z}[\zeta_q,x,t]$, $B(x,t) \equiv f(t)^r$ mod(x) for some $r \geq 1$ and $y^q - B(x,t)$ is irreducible over $\mathbb{Z}[[x]][\zeta_q,t]$.*

Proof: By Lemma 3.9 there is an irreducible q-cyclic Galois cover $\mathrm{Spec}(S) \to \mathbb{A}^1_{\mathbb{Z}[x]}$ whose fibre over (x), $\mathrm{Spec}(S/xS) \to \mathbb{A}^1_{\mathbb{Z}}$, is a connected mock cover ramified only at (f) such that pullback to the cover $\mathrm{Spec}(S') \to \mathbb{A}^1_{R'}$ has a corresponding extension of fraction fields $K'(t) \subset \mathrm{Frac}(S')$ given by a polynomial of the form $y^q - B(x,t)$ where $B(x,t) \in \mathbb{Z}[x,t]$ and $B(x,t) \equiv f(t)^r$ mod(x) for some $r \geq 1$ and $y^q - B(x,t)$ is irreducible over $\mathbb{Z}[[x]][\zeta_q,t]$.

Let X be the normalization of $\mathbb{P}^1_{\mathbb{Z}[x]}$ in $\mathrm{Spec}(S)$. By Lemma 3.10, $X \to \mathbb{P}^1_{\mathbb{Z}[x]}$ is unramifed at $t = \infty$. The fibre over (x) is a connected mock cover, since it is so generically.

To complete the proof we show that the pullback $\hat{X} \to \mathbb{P}^1{}_{\mathbb{Z}[[x]]}$ is regular and irreducible with a connected mock fibre over (x). The pullback of $X \to \mathbb{P}^1_{\mathbb{Z}[x]}$ to $\hat{X} \to \mathbb{P}^1_{\mathbb{Z}[[x]]}$ is irreducible since $y^q - B(x,t)$ is irreducible over $\mathbb{Z}[[x]][\zeta_q, t]$. It is locally irreducible since $\mathrm{Spec}(S) \to \mathbb{A}^1_{\mathbb{Z}[x]}$ is locally irreducible. The central fibre is a connected mock cover and is unramified at $t = 0$ since the fibres over $x = 0$ (which is the central fibre of $\hat{X} \to \mathbb{P}^1_{\mathbb{Z}[[x]]}$) of $\hat{X} \to \mathbb{P}^1_{\mathbb{Z}[[x]]}$ and $X \to \mathbb{P}^1_{\mathbb{Z}[x]}$ are the same and the fibre over (x) of the cover $X \to \mathbb{P}^1_{\mathbb{Z}[x]}$ is unramified at $t = 0$ since $f(0) \neq 0$. Thus any point of $\mathbb{P}^1_{\mathbb{Z}[[x]]}$ whose reduction mod (x) is (t) is unramified. It follows from Proposition 3.2 that the cover is regular. ∎

We next prove a proposition extending [H3, Prop. 2.2] which will allow us to patch these cyclic covers together. The key step in the proof, as in [H3], is Grothendieck's Existence Theorem [Gr2, Cor. 5.1.6]. That result asserts an equivalence of categories between formal sheaves of modules (Definition 2.19) and coherent sheaves of modules on a variety that is proper over an $\mathfrak{a}$-adically complete domain. Since an equivalence of categories preserves morphisms, it follows that there is a corresponding equivalence between coherent and formal sheaves of algebras, and similarly between covers and formal covers (viewed as the spectra of certain sheaves of algebras). In our situation, the cyclic covers, together with patching data on the central fibre, will define a formal sheaf of algebras, and as a result we will obtain a global G-Galois cover.

PROPOSITION 3.12: *Let G be a finite group and let $H_1, \ldots, H_r \subset G$ be subgroups which generate G. Let $\hat{R}$ be a normal domain that is complete with respect to a non-zero prime ideal $\mathfrak{a}$. Let Y be an integral scheme and let $Y \to \mathrm{Spec}(\hat{R})$ be a proper morphism of finite type. Define $Y_{\mathfrak{a}}$ to be the fibre of this morphism of over $\mathfrak{a}$, i.e. $Y_{\mathfrak{a}} = Y \times_{\mathrm{Spec}(\hat{R})} \mathrm{Spec}(\hat{R}/\mathfrak{a})$. Let $Z_i \to Y$ be a regular irreducible H_i-Galois cover with group H_i and branch locus L_i, whose central fibre $Z_{i,\mathfrak{a}} \to Y_{\mathfrak{a}}$ is a mock cover. Let $U_i = Y \smallsetminus \bigcup_{j \neq i} L_j$ and let $L_{i,\mathfrak{a}}$,$U_{i,\mathfrak{a}}$ be the central fibres of L_i,U_i. Suppose that $L_{1,\mathfrak{a}}$,..,$L_{r,\mathfrak{a}}$ are pairwise disjoint in $Y_{\mathfrak{a}}$.*

Then there exists a connected G-Galois mock cover $X_{\mathfrak{a}} \to Y_{\mathfrak{a}}$ whose inverse image over $U_{i,\mathfrak{a}}$ agrees, as a G-Galois cover, with the induced cover, $X_{i,\mathfrak{a}} = \mathrm{Ind}_{H_i}^G Z_{i,\mathfrak{a}} \to Y_{\mathfrak{a}}$ (Definition 2.21).

Moreover, there is a unique regular irreducible G-Galois cover $X \to Y$ whose central fibre is $X_{\mathfrak{a}}$ and which agrees with $X_i = \mathrm{Ind}_{H_i}^G Z_i$ over the formal completion (Definition 2.17) of U_i at $\mathfrak{a}$.

Proof: Form the G-Galois mock cover $X_{\mathfrak{a}} \to Y_{\mathfrak{a}}$ where the cover on the U_i is given by the restriction of $X_{i,\mathfrak{a}} \to Y_{\mathfrak{a}}$ and the patching on the overlap is is given by the G-labeling of the sheets. Let $U_{i,\mathfrak{a}^\nu} := U_i \times_{\mathrm{Spec}(\hat{R})} \mathrm{Spec}(\hat{R}/\mathfrak{a}^\nu)$. By Hensel's Lemma [B, III.4.3, Thm. 1], the cover X_i is trivial over the formal completion of $U = U_i \setminus L_i$ at $\mathfrak{a}$ for all i. For each i, U_i is an affine set, thus we may write $U_i = \mathrm{Spec}(E_i)$ for some ring E_i. Let $E_{i,\mathfrak{a}^\nu} = E_i \otimes_{\hat{R}} \hat{R}/\mathfrak{a}^\nu$. Let $Y_{\mathfrak{a}^\nu} = Y \times_{\hat{R}} \hat{R}/\mathfrak{a}^\nu$. Form the G-covers $X_{\mathfrak{a}^\nu} \to Y_{\mathfrak{a}^\nu}$, where the cover on (the affine subset) $U_{i,\mathfrak{a}^\nu}$ is given by $X_{i,\mathfrak{a}^\nu} = \mathrm{Spec}(E_{i,\mathfrak{a}^\nu}^{(G:H_i)})$ and the patching on the overlap is induced by the G-labeling on the fibre $X_{\mathfrak{a}}$. Let $\mathcal{E}_i$ be the inverse limit over ν of the $E_{i,\mathfrak{a}^\nu}$. Then the $\mathcal{E}_i^{(G:H_i)}$ define a coherent formal sheaf of G-algebras $\hat{\mathcal{E}}$ over the formal scheme $\hat{Y} := \varprojlim Y_{\mathfrak{a}^\nu}$. By Grothendieck's Existence Theorem [Gr2, Cor. 5.1.6], this is induced by a unique coherent sheaf of G-modules on Y, i.e. there exists a unique coherent sheaf of G-modules $\mathcal{E}$ such that $\hat{\mathcal{E}}$ is the formal completion of $\mathcal{E}$. By the comment preceding the proposition, $\mathcal{E}$ is canonically a sheaf of separable algebras and in fact defines a G-Galois cover. Namely, let $X = \underline{\mathrm{Spec}}(\mathcal{E})$. Then $X \to Y$ is a G-Galois cover whose mock central fibre is $X_{\mathfrak{a}}$ and which agrees with X_i over the formal completion of U_i at $\mathfrak{a}$.

To show X is unique, suppose $X' \to Y$ is a G-Galois cover with (mock) central fibre $X_{\mathfrak{a}}$ and suppose X' agrees with $X_i = \mathrm{Ind}_{H_i}^G Z_i$ over the formal completion of U_i at $\mathfrak{a}$. Then $X_{\mathfrak{a}^\nu}$ agrees with $X'_{\mathfrak{a}^\nu}$ over U_i. By Hensel's Lemma, $X_{\mathfrak{a}^\nu}$ and $X'_{\mathfrak{a}^\nu}$ are trivial away from the branch locus, i.e. the irreducible components of each are copies of the base labeled by the elements of G. Hence the G-labeling on $X_{\mathfrak{a}}$ gives the patching data on the overlaps $X_{\mathfrak{a}^\nu}|_{U_i \cap U_{i'}}$ and $X'_{\mathfrak{a}^\nu}|_{U_i \cap U_{i'}}$ for each ν. Therefore X and X' induce the same formal sheaf. It follows from the uniqueness assertion in Grothendieck's Existence Theorem that $X \cong X'$.

Now we proceed to show that X is irreducible. We do this by first proving the fibre $X_{\mathfrak{a}}$ is connected, then by proving X is connected.

Let $X'_{\mathfrak{a}}$ be the connected component of the G-Galois mock cover $X_{\mathfrak{a}}$ containing the identity sheet of $X_{\mathfrak{a}}$. In each induced cover $X_i = \mathrm{Ind}_{H_i}^G Z_i$, the sheets labeled by elements lying in the same coset gH_i lie in the same connected component of $X_{i,\mathfrak{a}}$. Therefore, if an irreducible sheet of $X_{\mathfrak{a}}$ labeled by some element g is in $X'_{\mathfrak{a}}$, then for $i = 1, \ldots, r$ and for all $h_i \in H_i$, the sheet labeled by gh_i is in $X'_{\mathfrak{a}}$. But G is generated by $H_1, \ldots, H_r$. Let $g' = h_1 h_2 \cdots h_r$. Now the sheet labeled by h_1 is in the same connected component as the identity sheet. By induction $g' = h_1 \cdots h_r$ lies in the same connected component as the identity sheet. It follows that every sheet of $X_{\mathfrak{a}}$ lies in $X'_{\mathfrak{a}}$ so we have shown $X_{\mathfrak{a}}$ is connected.

We may now show X is connected: Let X' be the connected component of X

containing $X_{\mathfrak{a}}$. Since $\hat{R}$ is complete with respect to $\mathfrak{a}$, $1-a$ is invertible for all $a \in \mathfrak{a}$. Hence $\mathfrak{a} \subseteq \mathrm{Jac}(\hat{R})$, the Jacobson radical of $\hat{R}$ [Mat, p. 10]. Since $\mathrm{Jac}(\hat{R})$ is the intersection of all maximal ideals $\mathfrak{m} \subset \hat{R}$, we have $\mathfrak{a} \subset \mathfrak{m}$. By hypothesis, Y is proper over $\mathrm{Spec}(\hat{R})$. By the definition of properness [Ha, p. 100], the image of any closed point is closed. Therefore all the closed points of Y are contained in $Y_{\mathfrak{a}}$. Since $X_{\mathfrak{a}} \subset X'$, the image of $X - X'$ contains no closed points of Y. Also, $X \to Y$ is a proper morphism, and the image of $X \smallsetminus X'$ is closed. Since this image contains no closed points, the image and hence $X \smallsetminus X'$ are empty and we have shown X is connected.

Observe that X is locally irreducible at the unramified points since X is étale over Y at these points. At the ramified points, by construction X is locally isomorphic to Z_i for some i and the Z_i are irreducible. Thus we have shown X is connected and locally irreducible; hence by Remark 2.8, X is irreducible. ∎

We now apply this patching result to the cyclic covers in Lemma 3.11. In order to do this first observe that there are an infinite number of non-constant pairwise relatively prime polynomials in $\mathbb{Z}[x]$, and we may choose a set of polynomials $\{f_i\}$ such that $f_i(0) \neq 0$ for each i. We can now show that we can realize all finite groups G as Galois groups of regular irreducible G-Galois covers of $\mathbb{P}^1_{\mathbb{Z}[[x]]}$.

THEOREM 3.13: *Let G be a finite group. Then there exists an irreducible G-Galois cover $X \to \mathbb{P}^1_{\mathbb{Z}[[x]]}$. Moreover, this cover can be chosen so that the central fibre is a mock cover, unramified at $t = 0$, the cover is regular and the branch locus is defined over $\mathbb{Z}$ (Definition 2.15).*

Proof: Let $g_1, \ldots, g_r$ be generators of G chosen so that the order of each g_i is a prime power, $p_i^{n_i}$. Choose r pairwise relatively prime non-constant polynomials $f_i(t) \in \mathbb{Z}[t]$ with $f_i(0) \neq 0$ for each i. For example, choose $f_i(t) = t - i$, $i = 1, \ldots, r$. By Lemma 3.11, there is an irreducible q_i-cyclic Galois cover $Z_i \to \mathbb{P}^1_{\mathbb{Z}[x]}$ whose central fibre is a connected mock cover ramified only at (f_i), and whose pullback to the completion, $\hat{Z}_i \to \mathbb{P}^1_{\mathbb{Z}[[x]]}$ is an irreducible cover (Definition 2.7). The cover $\hat{Z}_i \to \mathbb{P}^1_{\mathbb{Z}[[x]]}$ has the same central fibre as $Z_i \to \mathbb{P}^1_{\mathbb{Z}[x]}$, so the central fibre of $\hat{Z}_i \to \mathbb{P}^1_{\mathbb{Z}}$ is a connected mock cover unramified over $t = 0$ since the $\{f_i\}$ were chosen to be non-vanishing at $t = 0$. The branch loci of these covers $\hat{Z}_i \to \mathbb{P}^1_{\mathbb{Z}[[x]]}$ were chosen to be disjoint on the central fibre. We apply Proposition 3.12 to obtain a regular irreducible G-Galois cover of $\mathbb{P}^1_{\mathbb{Z}[[x]]}$ which is mock on the central fibre and unramified on the central fibre at $t = 0$. Thus any point of $\mathbb{P}^1_{\mathbb{Z}[[x]]}$ whose reduction mod (x) is in $V(t)$ is unramified. It follows from Proposition 3.2 that the cover is regular. ∎

To extend Theorem 3.13 to an arbitrary complete domain, we will need the following lemma and results from [Ha3].

LEMMA 3.14: *If $\hat{R}$ is an integral domain which is complete with respect to a non-zero prime ideal $\mathfrak{a}$ then $\hat{R}$ contains either a complete discrete valuation ring or a copy of $\mathbb{Z}[[x]]$. This subring may be chosen so that, in the first case, the contraction of $\mathfrak{a}$ is the maximal ideal and in the second case the contraction of $\mathfrak{a}$ is the ideal generated by x.*

Proof: We adapt an idea from the proof of [Ja, Lemma 1.5] to find the desired subring. First suppose that $\mathrm{char}(\hat{R}) = 0$. Then $\mathbb{Z} \subset \hat{R}$ and there are two cases:

CASE 1: $\mathbb{Z} \cap \mathfrak{a} \neq (0)$. Then $\mathbb{Z} \cap \mathfrak{a} = p\mathbb{Z}$ for some prime number p. Since $\hat{R}$ is complete with respect to $\mathfrak{a}$ and $p\mathbb{Z} \subset \mathfrak{a}$, we have $\mathbb{Z}_p \subseteq \hat{R}$ and $\mathfrak{a} \cap \mathbb{Z}_p = (p)$. Thus we take the desired subring to be $\mathbb{Z}_p$.

CASE 2: $\mathbb{Z} \cap \mathfrak{a} = (0)$. Since $\mathfrak{a}$ is a non-zero ideal, there exists some non-zero $x \in \mathfrak{a}$. If x were algebraic over $\mathbb{Q}$, then $a_n x^n + \cdots + a_1 x + a_0 = 0$ for some $a_0, a_1, \ldots, a_n \in \mathbb{Z}$ with $a_0 \neq 0$. Hence we would have $a_0 \in \mathbb{Z} \cap \mathfrak{a} = (0)$. From this contradiction we conclude that x is transcendental over $\mathbb{Q}$ and the polynomial ring $\mathbb{Z}[x]$ is contained in $\hat{R}$. Since $\hat{R}$ is complete with respect to $\mathfrak{a}$ and since $x \in \mathfrak{a}$, we have $\mathbb{Z}[[x]] \subseteq \hat{R}$. So $(x) \subseteq \mathbb{Z}[[x]] \cap \mathfrak{a}$. Let $z \in \mathfrak{a} \cap \mathbb{Z}[[x]]$. Then z is a power series in $\mathbb{Z}[[x]]$ without constant term (or else, by substracting off a multiple of x, we would have a non-zero element in $\mathbb{Z} \cap \mathfrak{a}$). Hence $z \in (x)$, so we have shown $\mathfrak{a} \cap \mathbb{Z}[[x]] = (x)$. Thus we may take the desired subring to be $\mathbb{Z}[[x]]$.

Now suppose $\mathrm{char}(\hat{R}) = p \neq 0$. Then $\mathbb{F}_p \cap \mathfrak{a} = 0$. As in Case 2, with $\mathbb{F}_p$ instead of $\mathbb{Z}$ and $\mathbb{Q}$, we show that we may take the desired subring to be $\mathbb{F}_p[[x]]$. ∎

Using the above lemma, Theorem 3.13 and [H3, Theorem 2.3], we may now generalize [H3, Theorem 2.3] to any domain which contains a domain that is complete with respect to a non-zero prime ideal. In particular, we may eliminate the hypothesis from [H3, Theorem 2.3] requiring the domain be normal and complete with respect to a maximal ideal and may instead consider domains complete with respect to any non-zero prime ideal.

Remark 3.15: It follows from the proof of [H3, Theorem 2.3] that one may choose the covers in the theorem to be regular and to be unramified at $t = 0$ on the central fibre.

Remark 3.16: Harbater does not explicitly state that the domain $\hat{R}$ must be normal in his hypothesis. However, this is implicit in the proof, as it relies on

[H3, Proposition Lemma 2.1], which does include normality as a hypothesis on $\hat{R}$.

THEOREM 3.17: *Let $\hat{R}$ be a domain containing a domain that is complete at a non-zero prime ideal and let G be a finite group. Then there exists a regular irreducible G-Galois cover $X \to \mathbb{P}^1_{\hat{R}}$. Moreover, this cover can be chosen so that the central fibre is a mock cover.*

Proof: By Lemma 3.14, $\hat{R}$ contains a subring $\hat{R}_0$ which is either a complete discrete valuation ring or a copy of $\mathbb{Z}[[x]]$. By [H3, Theorem 2.3] and Theorem 3.13 respectively, there is a regular irreducible G-Galois cover $X_0 \to \mathbb{P}^1_{\hat{R}_0}$. Let X be the pullback of X over $\mathbb{P}^1_{\hat{R}}$. Then the central fibre of $X \to \mathbb{P}^1_{\hat{R}}$ is a mock cover. The regularity implies that $X \to \mathbb{P}^1_{\hat{R}}$ is a regular irreducible G-Galois cover. ∎

By taking the generic fibre of the above cover we have the following corollary:

COROLLARY 3.18: *Let $\hat{R}$ be a domain which is not a field, such that $\hat{R}$ contains a subdomain that is complete with respect to a non-zero prime ideal. Then every finite group G is regular over the fraction field of $\hat{R}[x]$ with a rational point. Moreover, the cover corresponding to the extension may be chosen to be the generic fibre of a cover that is mock on the central fibre.*

We will need the following lemma in the proof of the next corollary.

LEMMA 3.19: *The fraction field of a Noetherian domain of dimension at least two is separably Hilbertian.*

Proof: Let R be a Noetherian domain of dimension at least two. Let R' be the integral closure of R. It follows from [ZS, Corollary to Thm. V.2.3] that R and R' have the same Krull dimension. It follows from [N, Theorem 33.10], that R' is a Krull domain. Hence by a theorem of Weissauer [FrJ, Thm. 14.17], the fraction field of R is separably Hilbertian. ∎

COROLLARY 3.20: *Let G be a finite group; let $\hat{R}$ be a Noetherian domain of dimension at least two that contains a subdomain that is complete with respect to a non-zero prime ideal. Then G is realizable over $F = \mathrm{Frac}(\hat{R})$.*

Proof: By Corollary 3.18 there exists an irreducible G-Galois cover $X \to \mathbb{P}^1{}_F$. By Lemma 3.19, F is separably Hilbertian. Therefore a suitable specialization of $X \to \mathbb{P}^1_F$ gives a Galois extension L/F with group G. ∎

As an immediate consequence we have an answer in the Noetherian case, to a question posed by Jarden [Ja, Problem 3.8].

COROLLARY 3.21: *Let G be a finite group, let R be a Noetherian domain and let $\mathbf{x} = x_1, \ldots, x_r$ with $r \geq 1$. If $\dim(R) \geq 1$ or $r \geq 2$ then G is realizable over the fraction field F of $R[[\mathbf{x}]]$.*

Proof: It follows from [B, III, 2.10, Cor. 6 to Thm. 2] that $R[[\mathbf{x}]]$ is a Noetherian domain. Since $R[[\mathbf{x}]]$ is complete at $(x_1 \cdots x_r)$ and the dimension of this domain is $\dim(R) + r \geq 2$, the result follows immediately from Corollary 3.20. ∎

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